Jordan-Hölder theory for complementary polyhedra

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For a Noetherian, Artinian object A inside an Abelian category $\mathcal{A}$, there exists a **Jordan-Hölder (JH) filtration**: [SP, 0FCD]:

$$0 = A_0 \subset \cdots \subset A_l = A$$

whose quotients $B_i := (A_i/A_{i-1})$ are simple, and whose **associated graded** $\bigoplus_{i=0}^{l} B_i$ is unique. Seshadri applied this to **slope-semistable vector bundles over a smooth projective K-curve X.**

JH filtrations of vector bundles [Ses67, Proposition 3.1.]

For a semistable vector bundle E over a compact Riemann surface X, there exists a JH filtration of subbundles:

$$0 = E_0 \subset \cdots \subset E_l = E$$

whose quotients $F_i := E_i/E_{i-1}$ are stable and have the same slope as E. Two JH filtrations have isomorphic associated gradeds $gr(E) := \bigoplus_{i=0}^{l} F_i$.

When $\mathrm{gr}(E) \cong \mathrm{gr}(F)$, $E \sim_S F$ are **S-equivalent**. For genus $g(X) \geq 2$, S-equivalence classes of semistable bundles have a **coarse moduli space** $\mathrm{MBun}_{r,d}^{S-eq}$ [Ses67, Theorem 8.1.] that is a normal projective K-variety. This generalizes to **principal bundles** by Ramanathan in [Ram96, I. 3.12. Proposition and II. 5.9. Theorem].

Adding $char(K) = 0$, we have a **good moduli space** $\pi : \mathcal{M}\mathrm{Bun}_{r,d}^{sst} \to \mathrm{MBun}_{r,d}^{S-eq}$ sending E to its S-equivalence class represented by the polystable $\mathrm{gr}(E)$, following Alper in [Alp13, 13.]. Generalizing this, there are good moduli spaces of **parahoric torsors** [AHH23, Theorem 8.1] and **parahoric Higgs torsors** [Rég25, Theorem 5.26.] that give a notion of S-equivalence.

How to calculate this explicitly? Behrend in **"Semi-stability of reductive group schemes"** [Beh95] determines **Harder-Narasimhan filtrations/canonical reductions** through **complementary polyhedra**. Faces of Weyl chambers in root systems Φ are called **facets** P, and fundamental weights in facets are **vertices** $\mathrm{vert}(P)$.

Complementary polyhedra [Beh95, Definition 2.1]

For a root system $\Phi \subset (V, \langle\_,\_\rangle)$, a map $d : \mathfrak{W}_\Phi \to V^*$ from the Weyl chambers is a **complementary polyhedron** if:

(A1) $\forall \mathfrak{c}, \mathfrak{d} \in \mathfrak{W}_\Phi$ with a common vertex $\lambda \in \mathrm{vert}(\mathfrak{c}) \cap \mathrm{vert}(\mathfrak{d})$, we have $d(\mathfrak{c})(\lambda) = d(\mathfrak{d})(\lambda)$.

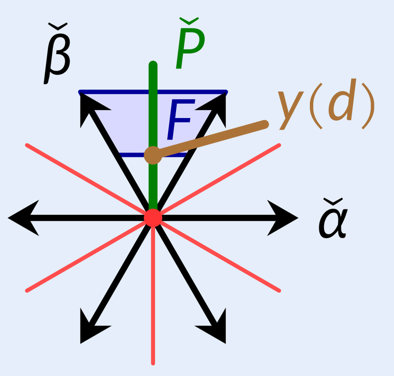
(A2) $\forall \mathfrak{c}, \mathfrak{d} \in \mathfrak{W}_\Phi$, and $\alpha \in \Phi$, such that $\mathfrak{c}$ is *α-conjugate* to $\mathfrak{d}$, i.e., $\mathfrak{c}$ is positive with respect to α, and $\mathfrak{d}$ is negative with respect to α, then $d(\mathfrak{c})(\alpha) \leq d(\mathfrak{d})(\alpha)$.

$F(P) := \mathrm{conv}\{d(\mathfrak{c}) \mid P \subset \bar{\mathfrak{c}}\}$ for a facet P, and $F := F(\{0\})$ is the *convex hull* of d.

For a principal G-bundle ξ, where G is reductive, the **automorphism group scheme** $\mathrm{Ad}(\xi)$ is reductive and is **rationally split**, i.e., splits over $\eta := \mathrm{Spec}(K(X))$. $\mathrm{Ad}(\xi)$ induces a complementary polyhedron d [Beh95, Proposition 6.6], such that the semi/poly/stability conditions of ξ, $\mathrm{Ad}(\xi)$ and d are equivalent. Facets P of Φ correspond to parabolic reductions of ξ [Beh95, Proposition 6.2], and canonical reductions correspond to the unique **special facet**, characterized by going through $\gamma(d) := \min(F)$ and $F(P)$ [Beh95, Lemma 3.11].

Complementary polyhedron example

Let $\Phi := A_2$ with simple roots $\triangle = \{\alpha, \beta\}$, label the Weyl chamber containing the root $\gamma \in A_2$ by $\mathfrak{c}_\gamma$. Define $d(\mathfrak{c}_{-\beta}) = d(\mathfrak{c}_\alpha) = \check{\beta}$ and $d(\mathfrak{c}_{-\alpha-\beta}) = d(\mathfrak{c}_{-\alpha}) = \widecheck{\alpha+\beta}$, $d(\mathfrak{c}_{\alpha+\beta}) = \check{\beta}/2$, $d(\mathfrak{c}_\beta) = (\widecheck{\alpha+\beta})/2$, which has the trapezium convex hull F and special facet P dual to $\check{P}$ (see right).



This story works not only for principal bundles (so also vector bundles), but also for **Higgs bundles** [Wiß14], **parabolic bundles** [HS10], and **parahoric (Higgs) torsors** [Hei17; Rég25]. It would be great to hijack this to construct **Jordan-Hölder filtrations/reductions** for these. For this, I define **Jordan-Hölder (JH) facets**.

Jordan-Hölder facets

For a semistable complementary polyhedron $d : \mathfrak{W}_\Phi \to V^*$, a facet P is **Jordan-Hölder (JH)** if:

(JH1) $\mathrm{span}(\check{P}) \cap F(P) = \{0\}$.

(JH2) $\mathrm{span}(\check{P}) \cap \mathrm{relint}(F(P)) \neq \emptyset$ and $\mathrm{span}(\check{P}, F(P)) = V^*$.

By an induction proof, I have verified that JH facets exist (future work).

Existence of Jordan-Hölder facets [Rot26]

For a semistable complementary polyhedron $d : \mathfrak{W}_\Phi \to V^*$, a Jordan-Hölder facet d exists.

I perform induction on the rank of Φ. The only rank 1 root system is A_1 with two Weyl chambers, so F becomes an interval and it is easy to check. The inductive step requires thinking about how d interacts with projections to hyperplanes.

JH facets P give stable **Levi-complementary polyhedra** $d_P : \mathfrak{W}_{\Phi_P} \to \check{P}^\perp$ of d, by looking at the definition of stability and applying (JH2). The analog of uniqueness of associated graded $\mathrm{gr}(E)$ is more involved and requires an induction proof.

Uniqueness of Levi of Jordan-Hölder facets [Rot26]

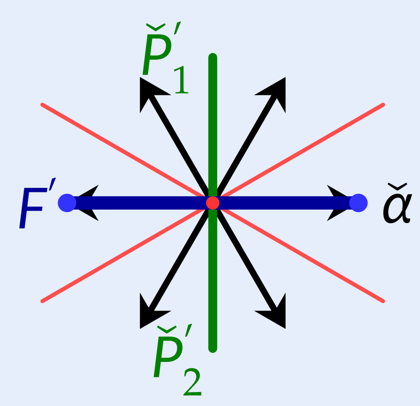
For a semistable complementary polyhedron $d : \mathfrak{W}_\Phi \to V^*$, and JH facets P_1 and P_2, then:

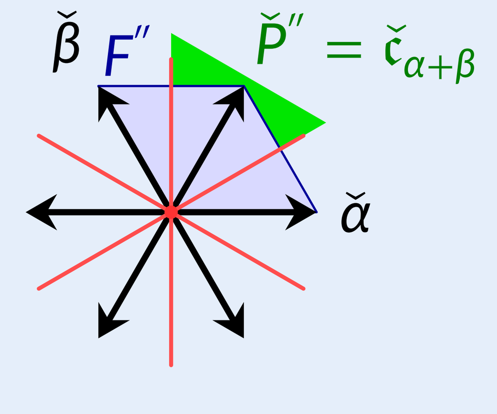
(i) $\exists \tau \in \mathrm{Aut}(\Phi)$, such that $\tau(P_1) = \tau(P_2)$.

(ii) $d_{P_1} = d_{P_2} : \mathfrak{W}_{\Phi_{P_1}} = \mathfrak{W}_{\Phi_{P_2}} \to P_1^\perp = P_2^\perp$.

Examples of Jordan-Hölder facets

Same setup as last example, we give two more complementary polyhedra: Let $d'(\mathfrak{c}_{-\beta}) = d'(\mathfrak{c}_\alpha) = d'(\mathfrak{c}_{\alpha+\beta}) = -\check{\alpha}$ and $d'(\mathfrak{c}_{-\alpha-\beta}) = d'(\mathfrak{c}_{-\alpha}) = d'(\mathfrak{c}_\beta) = \check{\alpha}$. Also let $d''(\mathfrak{c}_{-\beta}) = d''(\mathfrak{c}_\alpha) = \check{\beta}$, $d''(\mathfrak{c}_{-\alpha-\beta}) = \widecheck{\alpha+\beta}$, $d''(\mathfrak{c}_{-\alpha}) = d''(\mathfrak{c}_\beta) = \check{\alpha}$, and $d''(\mathfrak{c}_{\alpha+\beta}) = 0$. We have the convex hulls F' and F'' with JH facets:





For d' the τ in the prior theorem is flipping along the α-axis.

To recover Jordan-Hölder filtrations/reductions, we translate these results on complementary polyhedra to results on reductive group schemes.

Jordan-Hölder parabolics of reductive group schemes [Rot26]

Let $\mathbf{G}$ be a reductive group scheme over X that is rationally split, i.e., $\mathbf{G}$ splits over $\eta := \mathrm{Spec}(K(X))$. For a torus $\mathbf{T}_\eta$ of $\mathbf{G}_\eta$, let $d : \mathfrak{W}_{\Phi(\mathbf{G}_\eta, \mathbf{T}_\eta)} \to V^*_{\mathbf{G}_\eta, \mathbf{T}_\eta}$ be a semistable complementary polyhedron, then there exists Jordan-Hölder parabolics $\mathbf{P}$ of $\mathbf{G}$ with respect to d, unique up to Levi.

Like the Harder-Narasimhan/canonical reduction case, the complementary polyhedron d stores the stability of the moduli problem, and JH facets of d induce JH filtrations/reductions with the desired properties.

Examples of Jordan-Hölder facets inducing Jordan-Hölder filtrations

Let $E = L_1 \oplus F$ be the direct sum of a degree 1 line bundle L_1 and a rank 2 stable vector bundle F, which is the extension of $0 \to L_0 \to F \to L_2 \to 0$, $\deg(L_i) = i$. For $x \in X$, Serre vanishing gives $\mathrm{Ext}^1(L_2(-nx), L_0) = 0$ for $n \gg 0$, which implies $F \times_{L_2} L_2(-nx) \cong L_2(-nx) \oplus L_0$ so $L_2(-nx)$ embeds into F. Let $L' := L_2(-nx)^{sat}$ be the saturation, then for $\eta := \mathrm{Spec}(K(X))$ we have $E_\eta \cong (L_1)_\eta \oplus (L_0)_\eta \oplus (L')_\eta$. Using [Wiß14, 3.3.11 Remark.], the degrees of these bundles calculate the complementary polyhedron of $\mathrm{Ad}(E)$ to be d' from the last example. The JH filtrations:

$$0 \subset F \subset E \qquad\qquad 0 \subset L_1 \subset E$$

correspond to the JH facets of d'. For d'' the JH filtration is a full flag.

Jordan-Hölder filtrations of parahoric Higgs torsors [Rot26]

Let χ be an admissible stability parameter, and let ξ be a χ-semistable parahoric $\mathcal{G}$-torsor. $\exists$ JH reduction $s : X \to \xi/\mathcal{G}_{\mathrm{P}}$ to a parabolic subgroup $\mathcal{G}_{\mathrm{P}}$ of $\mathcal{G}$ such that:

(i) The reduction s is *admissible*, i.e., for every cocharacter $\boldsymbol{\lambda}_{\mathfrak{v}} : \mathbb{G}_{m,X} \to \mathrm{Ad}(s^*\xi)$ corresponding to a vertex $\mathfrak{v} \in \pi_0(t(\mathrm{Ad}(s^*\xi)))$, we have $n_\chi(\mathrm{Ad}(s^*\xi), \boldsymbol{\lambda}_{\mathfrak{v}}) = 0$.

(ii) The extension to the Levi $(s^*\xi)(\mathcal{L}_{\mathrm{P}})$ is $\chi|_{\mathcal{L}_{\mathrm{P}}}$-stable.

These reductions are unique up to Levi.

Inspired by the **quasi-JH filtrations** from Zhang in [Zha24], we have the moduli stack result.

Jordan-Hölder filtrations of moduli stacks [Rot26]

For a locally reductive algebraic K-stack $\mathcal{M}$, if $(d_x)_{x \in \mathcal{M}(K)}$ records the stability of x and each facet corresponds to a filtration $f : [\mathbb{A}_K^1/\mathbb{G}_{m,K}] \to \mathcal{M}$, then $\mathcal{M}$ has JH filtrations.

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