AGQ Group Project
Non-abelian Hodge Theory

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Abstract

We study the non-abelian Hodge decomposition of Higgs bundles (Chapter 3), which generalises the Hodge decomposition of complex manifolds (Chapter 1).

In order to construct this decomposition, we cover the Narasimhan-Seshadri correspondence between representations of fundamental groups and stable holomorphic vector bundles, and the Kobayashi-Hitchin correspondence between stable holomorphic vector bundles and vector bundles with Einstein-Hermitian metrics (Chapter 2).

Lastly, we present the isosingularity theorem of Simpson (Chapter 4), which is slightly stronger than the results usually understood as non-abelian Hodge theory. This result was used by Deligne to construct the twistor space of the hyperkähler manifold of Higgs bundles through λ-connections, which smoothly deform Higgs bundles and flat connections. The correspondence has also seen some recent applications, e.g. in Andrea Tirelli's paper on symplectic singularities of the moduli space of Higgs bundles.

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1 Introduction

1.1 Moduli theory of vector bundles

The non-abelian Hodge correspondence can be seen as a generalisation of a correspondence between moduli spaces of vector bundles and representations of fundamental groups. In this section, we review some results on moduli spaces of vector bundles.

In moduli theory one wants to gain insights on the geometry of constructions by studying their moduli spaces. Can we equip the set of holomorphic vector bundles on a curve with a holomorphic structure? Mumford in the 60s thought about this and developed *geometric invariant theory (GIT)* to answer such questions. He determined that, along with fixing discrete invariants such the rank r and degree d of a bundle, we need the notion of (semi)-stability to affirmatively answer this.

We recall the definitions of degree and stability for vector bundles.

Definition 1.1. For a holomorphic vector bundle E on a compact Riemann surface X:

(a) The *degree* of E is $\deg(E) = \langle c_1(E), [X] \rangle$, where $c_1(E)$ is the first Chern class of E and $[X]$ is the fundamental class of X.

(b) The *slope* of E is $\mu(E) = \deg(E)/\mathrm{rk}(E)$.

Definition 1.2. (a) A bundle E is *stable* if for all $F \subset E$ proper non-zero holomorphic subbundles:

$$\mu(F) < \mu(E).$$

(b) A bundle E is *semistable* if for all $F \subset E$ proper non-zero holomorphic subbundles:

$$\mu(F) \leq \mu(E).$$

(c) A bundle E is *polystable* if it is the direct sum of stable bundles of the same slope.

Let us look at the following moduli problems:

- $\textsc{VecBun}^{\mathrm{st}}_{r,d}(X)$: Stable holomorphic vector bundles on X of rank r and degree d.
- $\textsc{VecBun}^{\mathrm{sst}}_{r,d}(X)$: Semistable holomorphic vector bundles on X of rank r and degree d.

For a compact Riemann surface X of genus $g \geq 2$ (needed for Riemann-Roch calculations), we have the following results:

- [**Mum63**]: $\textsc{VecBun}^{\mathrm{st}}_{r,d}(X)$ has a good moduli space that is a complex manifold.
- [**Ses70**]: $\textsc{VecBun}^{\mathrm{sst}}_{r,d}(X)$ has a good moduli space that is a projective scheme over $\mathbb{C}$.
- [**NS65**]: The $\mathbb{C}$-points of $\textsc{VecBun}^{\mathrm{sst}}_{r,d}(X)$ form a good moduli space of polystable bundles of rank r and degree d (or S-equivalence classes of semistable bundles).
- [**New78**]: If rank and degree are coprime, the good moduli space is smooth and projective, and a fine moduli space up to scaling action (not true otherwise, see [**Ram73**]).

The constructions of these moduli spaces above are algebraic, and we would also like to have a more *analytic* or *gauge-theoretic* description of these spaces.

We first look at isomorphism classes of smooth vector bundles $\mathbb{E}$ on X of rank r. One can show that by perturbing sections of $\mathbb{E}$, $\mathbb{E}$ is smoothly isomorphic to $\det(\mathbb{E}) \oplus \mathbb{C}^{r-1}$, as seen in [**Tho23**, Proposition 7.2, Corollary 7.3]. Since complex line bundles are classified by their degree, there exists only *one* isomorphism class of smooth vector bundles of rank r and degree d. So for our purposes, we may fix a smooth vector bundle $\mathbb{E}$ of rank r and degree d. To construct $\textsc{VecBun}^{\mathrm{st}}_{r,d}(X)$ is then to construct holomorphic structures on $\mathbb{E}$.

Definition 1.3. *Holomorphic structures* are $\mathbb{C}$-linear operators:

$$\nabla^{0,1} : \Omega^0(X, \mathbb{E}) \to \Omega^{0,1}(X, \mathbb{E}) = \Omega^{0,1}(X, \mathbb{C}) \otimes_{C^\infty(X,\mathbb{C})} \Omega^0(X, \mathbb{E}),$$

satisfying the Leibniz rule for $f \in C^\infty(X, \mathbb{C})$, $s \in \Omega^0(X, \mathbb{E})$:

$$\nabla^{0,1}(fs) = \overline{\partial} f \otimes s + f\nabla^{0,1}(s).$$

If X is a complex analytic manifold of higher dimension, one must specify that holomorphic structures to fulfill an *integrability* condition $(\nabla^{0,1})^2 = 0$.

Remark 1.4. Holomorphic structures fully determine a holomorphic vector bundle E, up to isomorphism, whose underlying smooth vector bundle is $\mathbb{E}$, as seen in [**Tho23**, Proposition 7.8] and [**McC18**, Theorem 3.3.3].

Remark 1.5. By subtracting two holomorphic structures $\nabla^{0,1}_0, \nabla^{0,1}_1$ on $\mathbb{E}$, we get $C^\infty(X, \mathbb{C})$-linearity:

$$(\nabla^{0,1}_0 - \nabla^{0,1}_1)(fs) = f(\nabla^{0,1}_0 - \nabla^{0,1}_1)(s),$$

ensuring:

$$\begin{aligned}\nabla^{0,1}_0 - \nabla^{0,1}_1 \in \Omega^{0,1}(X, \mathrm{End}(\mathbb{E})) &= \Omega^{0,1}(X, \mathbb{C}) \otimes_{C^\infty(X,\mathbb{C})} \Omega^0(X, \mathrm{End}(\mathbb{E})), \\ &\cong \Omega^{0,1}(X, \mathbb{C}) \otimes_{C^\infty(X,\mathbb{C})} \Omega^0(X, \mathbb{E}) \otimes_{C^\infty(X,\mathbb{C})} \Omega^0(X, \mathbb{E}^*).\end{aligned}$$

Thus, the holomorphic structures form an infinite-dimensional complex affine space $\mathcal{A}$ modelled on $\Omega^{0,1}(X, \mathrm{End}(\mathbb{E}))$.

Let $\mathcal{G} = \mathrm{Aut}(\mathbb{E})$ be the Lie group of complex vector bundle automorphisms on $\mathbb{E}$, acting on $\mathcal{A}$ through conjugation:

$$g \in \mathcal{G} : \quad \nabla^{0,1} \mapsto g^{-1}\nabla^{0,1} g,$$

which are affine transformations on $\mathcal{A}$.

Takeaway: The moduli problem of $\textsc{VecBun}_{r,d}(X)$ is the same as that of $\mathcal{A}/\mathcal{G}$.

1.2 Classical Hodge theory

We set up and review the Hodge decomposition of Kähler manifolds. Non-abelian Hodge theory is motivated by this decomposition.

Let M be a compact complex manifold of complex dimension m. For a holomorphic chart $U \subset M$ with complex coordinates $z_1 = x_1 + iy_1, \dots, z_m = x_m + iy_m$, we have for $x \in U$:

$$T^*_x M = \mathrm{span}_{\mathbb{R}}(dx_j, dy_j), \qquad T^*_x M \otimes \mathbb{C} = \mathrm{span}_{\mathbb{C}}(dx_j, dy_j).$$

We define the complex-linear and complex-antilinear cotangent vectors:

$$\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j, \qquad \mathrm{d}\bar{z}_j = \mathrm{d}x_j - i\mathrm{d}y_j.$$

Using this, we can define:

$$T_x^*M^{1,0} = \mathrm{span}_{\mathbb{C}}(\mathrm{d}z_j), \qquad T_x^*M^{0,1} = \mathrm{span}_{\mathbb{C}}(\mathrm{d}\bar{z}_j).$$

Note that by the Cauchy-Riemann equations, this is independent of the choice of complex coordinates. This induces a splitting:

$$T_x^*M \otimes \mathbb{C} = T_x^*M^{1,0} \oplus T_x^*M^{0,1}.$$

This splitting can be done globally to give us a splitting of complex vector bundles:

$$T^*M \otimes \mathbb{C} = T^*M^{1,0} \oplus T^*M^{0,1}.$$

Using this, we get an induced splitting of exterior bundles:

$$\Lambda^r T^*M \otimes \mathbb{C} \cong \bigoplus_{l+n=r} \Lambda^l T^*M^{1,0} \otimes \Lambda^n T^*M^{0,1}.$$

Definition 1.6. A *differential-(l,n)-form* α on M is a smooth section of the vector bundle $\Lambda^l T^*M^{1,0} \otimes \Lambda^n T^*M^{0,1}$. The $C^\infty(M,\mathbb{C})$-module of differential-(l,n)-forms on M is denoted by $\Omega^{l,n}(M,\mathbb{C})$.

In local coordinates, we have:

$$\alpha = \sum_{|L|=l,|N|=n} a_{L,N} dz_L \wedge d\bar{z}_N, \qquad a_{L,N} \in C^\infty(U,\mathbb{C}).$$

Such forms account for the complex structure of M, for example, a smooth function $f \in C^\infty(M,\mathbb{C})$ is (anti)-holomorphic if and only if $df \in \Omega^{1,0}(M,\mathbb{C})$ ($df \in \Omega^{0,1}(M,\mathbb{C})$).

The $C^\infty(M,\mathbb{C})$-module $\Omega^r(M,\mathbb{C})$ of smooth differential-r-forms is equal to the direct sum $\bigoplus_{l+n=r} \Omega^{l,n}(M,\mathbb{C})$ of differential-(l,n)-forms. Hence, the (de Rham) exterior derivative:

$$d^r : \Omega^r(M,\mathbb{C}) = \bigoplus_{l+n=r} \Omega^{l,n}(M,\mathbb{C}) \to \Omega^{r+1}(M,\mathbb{C}) = \bigoplus_{l+n=r+1} \Omega^{l,r}(M,\mathbb{C}),$$

induces Dolbeault operators on the components:

$$\partial^{l,n} : \Omega^{l,n}(M,\mathbb{C}) \to \Omega^{l+1,n}(M,\mathbb{C}), \qquad \bar{\partial}^{l,n} : \Omega^{l,n}(M,\mathbb{C}) \to \Omega^{l+1,n}(M,\mathbb{C}).$$

Since $d = \partial + \bar{\partial}$, $d^2 = 0$ implies $\partial^2 = 0$ and $\bar{\partial}^2 = 0$, so ∂ and $\bar{\partial}$ defines complexes on differential-(l,n)-forms.

Definition 1.7. The group $H^{l,n}_{\mathrm{Dolb}}(M) = \ker\left(\bar{\partial}^{l,n}\right)/\mathrm{im}\left(\bar{\partial}^{l,n-1}\right)$ is the (l,n)-*Dolbeault cohomology group* of M.

We now require that M is a Kähler manifold, defined as follows:

Definition 1.8. Let M be a complex manifold of dimension m, then: (M,g,ω,I) is a *Kähler manifold*, where g is a Riemannian metric, I is a complex structure, and ω is a symplectic form, such that:

$$g(X,Y) = \omega(X,IY).$$

We often omit g and I and write (M,ω) for a Kähler manifold. We call ω the *Kähler form* of M.

The classical Hodge decomposition can be stated as follows for a Kähler manifold (M, ω):

Theorem 1.9 (Hodge theorem for Kähler manifolds [**CdS01**, Theorem 17.1])**.** *For a compact Kähler manifold (M, ω), there exists an isomorphism of cohomology groups as $\mathbb{C}$-vector spaces:*

$$H^r_{dR}(M, \mathbb{C}) \cong \bigoplus_{l+n=r} H^{l,n}_{Dolb}(M).$$

Furthermore, $H^{l,n}_{Dolb}(M)$ is finite-dimensional, and isomorphic to $H^{n,l}_{Dolb}(M)$ (canonically by complex conjugation).

Harmonic forms are used to prove this theorem, as discussed in Section 3.3. The theorem can be applied to the moduli problems in Section 1.1, as all Riemann surfaces are automatically Kähler.

Non-abelian Hodge theory (or non-abelian Hodge correspondence) will, in some sense, generalise Theorem 1.9 in the $r = 1$ case. We will go from de Rham cohomology with coefficients in $\mathbb{C}$ to Čech cohomology with coefficients in $\mathbf{GL}(r, \mathbb{C})$, which is a non-abelian group (hence the name of *non-abelian* Hodge correspondence). We will explain this further in Section 3.1.

Invoking Čech cohomology for $r = 1$ suggests that the correspondence says something about the moduli problem $\textsc{VecBun}^{\mathrm{st}}_{r,d}(X)$. In fact, the non-abelian Hodge correspondence is a generalisation of the Narasimhan-Seshadri correspondence from [**NS65**] discussed in the next chapter.

2 Narasimhan-Seshadri and Kobayashi-Hitchin correspondences

2.1 Narasimhan-Seshadri correspondence (of degree 0)

Let X be a compact Riemann surface of genus $g \geq 2$. We have the following correspondence:

Theorem 2.1 (Narasimhan-Seshadri correspondence, [**NS65**])**.** *There is a bijection between* $\textsc{VecBun}^{st}_{r,0}$*, and the moduli space of irreducible unitary representations of the fundamental group* $\pi_1(X)$*:*

$$\textsc{VecBun}^{st}_{r,0} \cong \mathit{Hom}^{irred}(\pi_1(X), \mathbf{U}(r))/\mathbf{U}(r).$$

Irreducible unitary representations in $\mathrm{Hom}^{\mathrm{irred}}(\pi_1(X), \mathbf{U}(r))$ are isomorphic to each other by conjugation by $\mathbf{U}(r)$, hence, we quotient out by $\mathbf{U}(r)$ to obtain the moduli space $\mathrm{Hom}^{\mathrm{irred}}(\pi_1(X), \mathbf{U}(r))/\mathbf{U}(r)$.

This correspondence is surprising, as we only need to know the underlying topology of X (so only the genus g) in order to classify stable holomorphic vector bundles on X. Using the group presentation of $\pi_1(X)$:

$$\pi_1(X) = \left\langle A_1, \ldots, A_g, B_1, \ldots, B_g \;\middle|\; \prod_i [A_i, B_i] = 1 \right\rangle,$$

the values $\rho(A_i)$ and $\rho(B_i)$ of a unitary representation ρ of $\pi_1(X)$ give the monodromy of a smooth vector bundle E along nontrivial loops in $\pi_1(X)$. Being a unitary representation, ρ induces a Hermitian metric h on the induced bundle E_ρ.

Remark 2.2. Narasimhan and Seshadri's proof used properties of algebraic varieties, Weil's 1938 paper on *abelian functions* in [**Wei38**], and Atiyah's classification of vector bundles over an elliptic curve from 1957 in [**Ati57**].

In this section, we present a proof strategy of this correspondence, but more from an analytic perspective following [**Tho23**] (originally from [**Don83**]).

Our comment earlier about monodromy refers to a flat connection ∇ on a smooth vector bundle E, which exists when E is of degree 0. The *Riemann-Hilbert correspondence* makes this precise:

Theorem 2.3 (Riemann-Hilbert correspondence)**.** *(i)* ***Real version*** *(see [**Tho23**, Theorem 3.4]): Let K be a compact Lie group. There exists a diffeomorphism:*

$$\mathit{Hom}(\pi_1(X), K)/K \cong \{\mathit{flat}\text{-}K\text{-}\mathit{connections}\}/\!\!/\mathcal{G},$$

where $\mathcal{G}$ consists of smooth gauge transformations acting on flat-K-connections.

(ii) ***Complex version*** *(see [**Sim94b**, Proposition 7.8]): Let G be a complex reductive group. There exists a complex analytic isomorphism:*

$$\mathit{Hom}(\pi_1(X), G)/G \cong \{\mathit{flat}\text{-}G\text{-}\mathit{connections}\}/\!\!/\mathcal{G}^{\mathbb{C}},$$

where $\mathcal{G}^{\mathbb{C}}$ consists of complex gauge transformations acting on flat-G-connections.

In the right-hand side, we have a Hamiltonian reduction, as seen in [**Tho23**, Theorem 5.1]. There is a symplectic structure on the space of $\mathbf{U}(r)$-connections (see Subsection 4.1.1 for details), that is compatible with the $\mathcal{G}$-action. One can calculate that the moment map μ of this action maps a connection to its curvature. Thus, $\mu^{-1}(0)$ consists of flat connections, and $\mu^{-1}(0)/\mathcal{G}$ equals the right-hand side.

The proof of Theorem 2.1 will also use a formal analogue of the *Kempf-Ness theorem*, which is set up as follows: Let G be a complex reductive group and let K be a maximal compact Lie subgroup. Let V be a finite-dimensional complex vector space for which K acts on V unitarily (with respect to a Hermitian inner product h on V). By complexification, this action lifts to a G-action on V.

Theorem 2.4 (Kempf-Ness theorem)**.** *Let $X \subset V$ be a G-invariant affine variety. The action of K on X is Hamiltonian with moment map μ, and the inclusion map $\mu^{-1}(0) \hookrightarrow X^{sst}$ induces a homeomorphism from the symplectic reduction $X/\!\!/K = \mu^{-1}(0)/K$ to the affine GIT quotient $X^{sst}/\!\!/G$.*

We now sketch a proof of Theorem 2.1:

Proof of Theorem 2.1. Using the Riemann-Hilbert correspondence in Theorem 2.3, we have the diffeomorphism:

$$\mathrm{Hom}(\pi_1(X), \mathbf{U}(r))/\mathbf{U}(r) \cong \{\text{flat-}\mathbf{U}(r)\text{-connections}\}/\!\!/\mathcal{G}.$$

Then, due to a formal analogue the Kempf-Ness theorem (Theorem 2.4), we also have a homeomorphism:

$$\{\text{flat-}\mathbf{U}(r)\text{-connections}\}/\!\!/\mathcal{G} \cong \{\text{flat-}\mathbf{U}(r)\text{-connections}\}^{\text{sst}}/\!\!/\mathcal{G}^{\mathbb{C}},$$

where $\mathcal{G}^{\mathbb{C}}$ is the complexification of the gauge group $\mathcal{G}$. The right-hand side is an infinite-dimensional analogue of a GIT-quotient, on the affine space of (GIT-semistable) flat-$\mathbf{U}(r)$-connections (see Remark 2.5).

A flat-$\mathbf{U}(r)$-connection can be written as $\nabla = d + A$ in terms of its *connection potential* $A \in \Omega^1(X, \mathrm{Ad}(X \times \mathbf{U}(r)))$, where $X \times \mathbf{U}(r)$ is the trivial-$\mathbf{U}(r)$-bundle. We ask how to write the $\mathcal{G}^{\mathbb{C}}$-action on $d + A$: Using a bundle-valued version of the Hodge decomposition in Theorem 1.9 (see [**Bra**, Theorem 3.12]), we can equivalently ask how this acts on components $A^{1,0}$ and $A^{0,1}$ of A. For $g \in \mathcal{G}^{\mathbb{C}}$, we have the affine transformation on $A^{0,1}$:

$$gA^{0,1} = gA^{0,1}g^{-1} + g\nabla^{0,1}g^{-1}, \tag{2.1}$$

where $\nabla^{0,1}$ denotes the trivial holomorphic structure on $X \times \mathbf{U}(r)$. This action uniquely extends to an action on $d + A$ such that $g(d + A)$ is a flat-$\mathbf{U}(r)$-connection. In this way, $A \mapsto A^{0,1}$ defines a bijection:

$$\{\text{flat-}\mathbf{U}(r)\text{-connections}\}^{\text{sst}}/\!\!/\mathcal{G}^{\mathbb{C}} \cong \Omega^{0,1,\text{sst}}(\mathfrak{gl}(r, \mathbb{C}))/\!\!/\mathcal{G}^{\mathbb{C}}, \tag{2.2}$$

where $\Omega^{0,1,\text{sst}}(\mathfrak{gl}(r, \mathbb{C}))$ consists of the semistable $0,1$-$\mathfrak{gl}(r, \mathbb{C})$-valued forms on X, quotiented by the $\mathcal{G}^{\mathbb{C}}$-action on $A^{0,1}$ from (2.1). From the takeaway of Section 1.1, the right-hand side of (2.2) embeds into $\mathrm{V{\scriptsize EC}B{\scriptsize UN}}_{r,0}$.

By restricting from $\mathrm{Hom}(\pi_1(X), \mathbf{U}(r))/\mathbf{U}(r)$ to irreducible representations, we obtain a bijection between $\mathrm{Hom}^{\text{irred}}(\pi_1(X), \mathbf{U}(r))/\mathbf{U}(r)$ and $\mathrm{V{\scriptsize EC}B{\scriptsize UN}}^{\text{st}}_{r,0}$. $\square$

Note that the Narasimhan-Seshadri correspondence does not claim anything more than a bijection, despite both spaces being smooth manifolds.

Remark 2.5. We have glossed over some details in the proof sketch:

- We did not say precisely how the irreducibility of the unitary representations correspond to the slope-stability of vector bundles. For a detailed explanation, look at [**WG08**, Appendix, Section 2.2, Theorem 2.6-2.7].
- The GIT-quotient side of Kempf-Ness in this application does not make sense, since we are quotienting by infinite-dimensional spaces. Donaldson in [**Don83**] follows the proof idea of Kempf-Ness by showing that the complex gauge orbit of a unitary connection intersects at the zero set of the moment map (which is curvature, so the zero set is the flatness condition). He showed this using a gradient-descent method on the *Yang-Mills functional*:
$$A \mapsto ||F(A)||^2,$$
where the norm is an L_2-norm (see [**AB83**, page 548]).

  If we start at a point in the complex gauge orbit, the gradient flow gives us a sequence of connections in the same complex gauge orbit. This sequence converges to an absolute minimum (flat connection) if and only if our orbit is GIT-semistable.

  To prove these claims, one also needs the *Uhlenbeck-Yau compactness theorem*.

Note that this correspondence can be extended from vector bundles to principal-G-bundles, see Ramanathan's result in [**Ram73**, Theorem 7.1].

2.2 Narasimhan-Seshadri correspondence (of any degree)

We mention a generalisation of the Narasimhan-Seshadri correspondence for any degree d, with the initial case being $d = 0$, following [**WG08**, Appendix, Section 4].

On the right side of the correspondence in Theorem 2.1, the fundamental group $\pi_1(X)$ is insufficient as our vector bundles have nontrivial *topological type* not seen in $\pi_1(X)$. To fix this, we need universal central extensions in Γ of $\pi_1(X)$. We look at the short exact sequence of groups:
$$0 \to \mathbb{Z} \to \Gamma \to \pi_1(X) \to 0. \tag{2.3}$$
where $1 \in \mathbb{Z}$ maps to a central element $J \in \Gamma$. The group Γ has the presentation:
$$\Gamma = \left\langle A_1, \ldots, A_g, B_1, \ldots, B_g, C | \prod_i [A_i, B_i] C = 1 \right\rangle,$$

Using (2.3), any representation $\tilde{\rho} : \pi_1(X) \to \mathbf{PGL}(r, \mathbb{C}) = \mathbf{GL}(r, \mathbb{C})/\mathbb{C}^*$ lifts to a representation $\rho : \Gamma \to \mathbf{GL}(r, \mathbb{C})$. The uniformisation theorem ensures that X has a universal conformal covering of the upper half plane $\mathbb{H}$, for which $\mathbf{PGL}(r, \mathbb{C})$ acts by Möbius transformations. From the projective bundle $\hat{P} = (\mathbb{H} \times_{\tilde{\rho}} \mathbb{P}(\mathbb{C}^r))$ on X, we find a vector bundle E_ρ on X whose extension from $\mathbf{GL}(r, \mathbb{C})$ to $\mathbf{PGL}(r, \mathbb{C})$ is $\hat{P}$.

There exists a connection ∇ on E_ρ with *constant central curvature* (see [**WG08**, Appendix, Section 2.2]) such that at any fibre $x \in X$, $\tilde{\rho} : \pi_1(X, x) \to \mathbf{PGL}((E_\rho)_x)$ is the monodromy of ∇ at x. From this, we can define a subgroup of $\mathrm{Hom}(\Gamma, \mathbf{GL}(r, \mathbb{C}))$:
$$\mathrm{Hom}_d(\Gamma, \mathbf{GL}(r, \mathbb{C})) = \{\rho \in \mathrm{Hom}(\Gamma, \mathbf{GL}(r, \mathbb{C})) | c_1(E_\rho) = d\},$$
and so we get a more general *Narasimhan-Seshadri correspondence* for degree d:
$$\textsc{VecBun}^{\mathrm{st}}_{r,d} \cong \mathrm{Hom}^{\mathrm{irred}}_d(\pi_1(X), \mathbf{U}(r))/\mathbf{U}(r).$$

2.3 Kobayashi-Hitchin correspondence

Given a holomorphic vector bundle E on X, the space of all Hermitian metrics on E is an infinite-dimensional space. One can then ask, is there a canonical choice of metric on E? One such choice is the following:

Definition 2.6. Let h be a Hermitian metric on a holomorphic vector bundle E. Then h is *Hermite-Einstein* if there exists a constant c such that:

$$\Lambda_\omega(iF_h) = cI_E.$$

Here, I_E is the identity map on E, F_h is the curvature of the associated Chern connection, and Λ_ω is the contraction operator with a fixed Kähler form ω on X.

Remark 2.7. Many sources will refer to a *constant central curvature metric*, as mentioned in the previous subsection. On a curve, these notions agree, although the constants may differ. In higher dimensions, both to define stability, as well as in the definition of a Hermite-Einstein metric, one needs to choose a polarisation (i.e. a Kähler metric).

The constant c in the above definition is topological. By Chern-Weil theory, it is given by:

$$c = 2\pi\mu(E).$$

Theorem 2.8 (Kobayashi-Hitchin correspondence)**.** *Let E be an indecomposable holomorphic vector bundle on X. Then E is stable if and only if it admits a Hermite-Einstein metric.*

Recall that the Chern connection is determined by a Hermitian metric and a holomorphic structure as defined above. We have taken the approach of viewing the holomorphic structure as fixed, and varying the metric. One can also fix a metric, and consider the set of all holomorphic structures instead. Because of this, the equation:

$$\Lambda(iF_h) = cI_E,$$

is sometimes also called the *Hermitian-Yang-Mills equation.* Using this, we can then make formal analogies with GIT:

- The group $\mathcal{G} = \mathrm{Aut}(\mathbb{E})$ acts on the space of holomorphic structures $\mathcal{A}$ on the fixed underlying smooth bundle $\mathbb{E}$, as in Remark 1.5.
- The subgroup $\mathcal{K} = \mathbf{U}(\mathbb{E}, h)$ of (fibrewise) unitary automorphisms is the corresponding maximal compact subgroup.
- The operator $\Lambda(iF_h) - cI_E$ defines a moment map for the $\mathcal{K}$-action on the space of holomorphic structures.

Using this, Kempf-Ness would suggest a link between Hermite-Einstein metrics and an algebro-geometric stability condition, which is what is predicted by the Kobayashi-Hitchin correspondence. However, in Kempf-Ness, the corresponding stability condition to a zero of a moment map is *polystability*, from Definition 1.2:

Definition 2.9. A bundle E is *polystable* if it is a direct sum of stable bundles of the same slope.

Using this, we have the following:

Theorem 2.10 (Kobayashi-Hitchin correspondence)**.** *Let E be a holomorphic vector bundle on X. Then E is polystable if and only if it admits a Hermite-Einstein metric.*

Remark 2.11. The Kobayashi-Hitchin correspondence holds more generally. Donaldson proved the result for X being projective, and Uhlenbeck-Yau proved the result for X being compact and Kähler. In the higher-dimensional cases, one needs to also test stability against *subsheaves* and not just subbundles.

The proof that E admitting a Hermite-Einstein metric implies polystability follows from a direct computation. Namely, E is semistable, and one needs to show that if $\mathcal{F} \subseteq E$ is a coherent subsheaf of the same slope, then $\mathcal{F}$ is in fact a subbundle and $E \cong \mathcal{F} \oplus E/\mathcal{F}$.

For the converse, Uhlenbeck-Yau considers a one-parameter family $L_\epsilon = 0$ of differential equations, where we have a dichotomy:

- either there exists a solution to $L_0 = 0$ which is the Hermite-Einstein equation,
- or we can construct a destabilising subsheaf of E.

2.4 Relation between the Narasimhan-Seshadri and Kobayashi-Hitchin correspondences

To conclude this chapter, we will relate the Kobayashi-Hitchin correspondence to the Narasimhan-Seshadri correspondence. The most straightforward generalisation is when $\deg(E) = 0$, where we have the following:

Corollary 2.12. *Let E be an irreducible holomorphic vector bundle on X of degree 0, then E is stable if and only if it admits a flat metric.*

The Riemann-Hilbert correspondence relates the set of flat bundles (modulo gauge) and the set of representations $\pi_1(X) \to \mathbf{U}(r)$. Combining these two, we recover the Narasimhan-Seshadri correspondence in degree 0.

Next, we would like to generalise this to arbitrary degree, as we did with the Narasimhan-Seshadri correspondence in Section 2.2. To do this, we will first need to pass from vector bundles to principal bundles. The Riemann-Hilbert correspondence generalises naturally to the following:

Theorem 2.13 ([**Kob87**, Proposition I.2.6])**.** *Let P be a principal-G-bundle over X, then P admits a flat connection if and only if P is defined by a representation $\pi_1(X) \to G$.*

Taking $G = \mathbf{GL}(r, \mathbb{C})$ recovers the statement for vector bundles. In this case, we are interested in the case of $G = \mathbf{PGL}(r, \mathbb{C})$. Let P be a principal-$\mathbf{GL}(r, \mathbb{C})$-bundle over X. Then $\hat{P} = P/\mathbb{C}^*$ is a principal-$\mathbf{PGL}(r, \mathbb{C})$-bundle.

Definition 2.14. A connection ∇ on a vector bundle E is *projectively flat* if the induced connection on $\hat{P}$ is flat, where P is the associated principal-$\mathbf{GL}(r, \mathbb{C})$-bundle.

Specialising the theorem to the case of $G = \mathbf{PGL}(r, \mathbb{C})$, we obtain:

Corollary 2.15 ([**Kob87**, Corollary I.2.7])**.** *Let E be a complex vector bundle over X, then E admits a projectively flat connection if and only if the associated $\mathbf{PGL}(r, \mathbb{C})$-bundle $\hat{P}$ is given by a representation $\pi_1(X) \to \mathbf{PGL}(r, \mathbb{C})$.*

Finally, we will connect this to the Kobayashi-Hitchin correspondence as stated above:

Lemma 2.16 ([**Kob87**, Proposition I.2.8])**.** *A connection ∇ on a complex vector bundle E is projectively flat if and only if there exists a complex valued 2-form α such that:*

$$F_\nabla = \alpha I_E.$$

Proof. Let $\mu : \mathbf{GL}(r,\mathbb{C}) \to \mathbf{PGL}(r,\mathbb{C})$ denote the quotient map, and $\mu' : \mathfrak{gl}(r,\mathbb{C}) \to \mathfrak{pgl}(r,\mathbb{C})$ the induced map on Lie algebras. If F is the curvature of the connection D on E, then the curvature of the connection on $\hat{P}$ is given by $\mu'(F)$. The kernel of μ' is exactly the scalar multiples of the identity. □

On a curve, the condition that $F_\nabla = \alpha I_E$ is exactly the Hermite-Einstein equation. Thus, we obtain a correspondence between projectively flat connections and Hermite-Einstein metrics, which in turn, by the Kobayashi-Hitchin correspondence, corresponds to polystable vector bundles. To be precise, we have the following theorem.

Theorem 2.17 ([**Kob87**, Theorem V.2.7])**.** *Let E be a holomorphic vector bundle over a curve X. The following are equivalent:*

(i) E is stable.

(ii) The associated projective bundle $\mathbb{P}(E)$ comes from an irreducible representation $\pi_1(X) \to \mathbf{PU}(r)$.

(iii) E admits a projectively flat Hermitian structure.

This theorem can be thought of as a restatement of the Narasimhan-Seshadri theorem, see [**AB83**]. On the other hand, using Lemma 2.16, the theorem implies that any stable bundle admits a *weak Hermite-Einstein metric*. That is, we have a smooth function f such that:

$$\Lambda(iF_h) = fI_E.$$

By considering $e^\phi h$ for some smooth function ϕ, one can show that if we can solve the weak Hermite-Einstein equation, then we can solve the Hermite-Einstein equation. Thus, we have recovered Theorem 2.8.

3 Non-abelian Hodge theory

In this section, we will generalise the Narasimhan-Seshadri correspondence from Theorem 2.1 by identifying irreducible representations with Higgs bundles. By dropping the unitary requirement, vector bundles do not suffice, leading to the notion of Higgs bundles.

As before, we fix a compact Riemann surface X of genus $g \geq 2$.

3.1 From abelian to non-abelian

From the Hodge decomposition in Theorem 1.9, we get that:

$$\begin{aligned} H^1(X, \mathbb{C}) &= H^{1,0}(X) \oplus H^{0,1}(X), \\ &= H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X). \end{aligned}$$

We want to replace $\mathbb{C}$ by $\mathbf{GL}(r, \mathbb{C})$. The two corresponding terms on the right-hand side which we get are then given by:

$$\check{H}^1(X, \mathbf{GL}(r, \mathbb{C})) \quad \text{and} \quad H^0(X, \mathbf{GL}(r, \mathbb{C}) \otimes \Omega^1_X).$$

The first term corresponds to a holomorphic vector bundle, and the second corresponds to a Higgs bundle on it, which we will define later.

To motivate why we would want to consider this, when $r = 1$, we have $\mathbf{GL}(r, \mathbb{C}) = \mathbb{C}^*$, and we have that:

$$H^1(X, \mathbf{GL}(1, \mathbb{C})) = H^1(X, \mathbb{C}^*) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*),$$

which corresponds to one-dimensional complex representations of $\pi_1(X)$. To make this a bit more precise, we will consider the following moduli spaces as defined in [**Sim94a**]:

- the *Betti moduli space* $\mathcal{M}^B_r(X)$ of representations of $\pi_1(X)$ of rank n,
- the *de Rham moduli space* $\mathcal{M}^{dR}_{r,0}(X)$ of flat connections of rank n,
- the *Dolbeault moduli space* $\mathcal{M}^H_{r,0}(X)$ of Higgs bundles of rank n.

By the Riemann-Hilbert correspondence in Theorem 2.3, the first two are complex analytically isomorphic. The non-abelian Hodge correspondence then states that these are in bijection with the Dolbeault moduli space. In fact, more is true, which will be covered in Chapter 4.

In this chapter, we will define the objects involved, and make more precise statements about the correspondence, as well as some of the ideas involved in the proof.

3.2 Higgs bundles

Definition 3.1. A *Higgs bundle* (E, Φ) is a holomorphic vector bundle E, with a *Higgs field* $\Phi \in H^0(X, \mathrm{End}(E) \otimes \Omega^1_X)$.

Let us now describe how to arrive at this definition from the perspective of deformation theory. From before, we have seen that the space of holomorphic structures on a fixed smooth vector bundle E is an affine space modelled on $\Omega^{0,1}(X, \mathrm{End}(E))$. Since $\Omega^{0,2}(X, \mathbb{C}) = 0$, given $A \in \Omega^{0,1}(X, \mathrm{End}(E))$, and a fixed holomorphic structure $\nabla^{0,1}$ on E, we must have that $\nabla^{0,1} A = 0$. Thus, we get a class $[A] \in H^{0,1}(X, \mathrm{End}(E, \nabla^{0,1}))$.

First of all, note that

$$H^{0,1}(X, \mathrm{End}(E, \nabla^{0,1})) = H^1(X, \mathrm{End}(E, \nabla^{0,1}))$$

The left hand side is Dolbeault cohomology, and the right hand side is sheaf cohomology. Note that when studying deformations, we often see the right hand side more. By Serre duality, we then obtain that

$$H^1(X, \mathrm{End}(E, \nabla^{0,1})) \cong H^0(X, \mathrm{End}(E, \nabla^{0,1})^* \otimes \Omega^1_X)^* \cong H^0(X, \mathrm{End}(E) \otimes \Omega^1_X).$$

With this, we can think of a Higgs bundle as a holomorphic vector bundle along with a deformation of the complex structure.

Remark 3.2. Even though we will only work with X being a curve, it is interesting to consider what happens over general compact Kähler manifolds. In this case, we will need an additional *integrability* condition in the definition of a Higgs bundle. That is, we would like $\nabla^{0,1} + \Phi$ to define a holomorphic structure on E, and so we will need it to satisfy the integrability condition:

$$(\nabla^{0,1} + \Phi)^2 = 0.$$

Since $(\nabla^{0,1})^2 = 0$, and $\nabla^{0,1}\Phi = 0$ as Φ is holomorphic, the integrability condition is then given by:

$$\Phi \wedge \Phi = 0.$$

In local coordinates $z_1, \ldots, z_n$, we can write:

$$\Phi = \sum_i \Phi_i \mathrm{d}z_i,$$

and the integrability condition becomes $[\Phi_i, \Phi_j] = 0$.

Next, we would like to describe the Dolbeault moduli space of Higgs bundles. As usual, this means that we will need a slope stability condition. As before, the slope of a vector bundle E is:

$$\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}.$$

Definition 3.3. A Higgs bundle (E, Φ) is *stable* (resp. *semistable*) if for all subbundles $F \subseteq E$ which are Φ-invariant, we have that $\mu(F) < \mu(E)$ (resp. $\mu(F) \le \mu(E)$).

In the above definition, a subbundle F is Φ-invariant if for all vector fields X, we have that $\Phi(X)(F) \subseteq F$. Equivalently, this means that Φ restricts to a section of $\mathrm{End}(F) \otimes \Omega^1_X$. In terms of deformation theory, this means that the deformation of E induces a deformation of F.

Example 3.4. Suppose E is a stable (resp. semistable) vector bundle. Then for any Higgs field Φ, (E, Φ) is stable (resp. semistable).

Using this, we can now state the non-abelian Hodge correspondence:

Theorem 3.5 (Non-abelian Hodge correspondence)**.** *There exists a bijection:*

$$\mathcal{M}^{H,pst}_{r,0}(X) \cong \mathit{Hom}(\pi_1(X), \mathbf{GL}(r, \mathbb{C}))/\mathbf{GL}(r, \mathbb{C})$$

between the moduli space of polystable degree 0 Higgs bundles and the $\mathbf{GL}(n, \mathbb{C})$*-character variety.*

In terms of the moduli spaces, this relates the Dolbeault and Betti moduli spaces.

Remark 3.6. In fact, the above correspondence is a homeomorphism, which is given by the holonomy map. This is seen in [**Sim92**, Proposition 1.5]. More is true, this is a diffeomorphism according to [**Boa12**]. We will also justify this in Chapter 4.

3.3 Harmonic flat bundles

To start off, we will recall the proof of the Hodge theorem for Riemannian manifolds. Let (M^n, g) be a compact Riemannian manifold. We can define the *codifferential* operator $\mathrm{d}^* : \Omega^k(M) \to \Omega^{k-1}(M)$ as the L^2-adjoint to the exterior derivative d. More concretely, in terms of the Hodge star operator $\star$, we have that:

$$\mathrm{d}^* = (-1)^k \star^{-1} \mathrm{d}\star$$

Using this, we can define a Laplacian:

$$\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}.$$

This turns out to be a second-order linear self-adjoint elliptic operator, and the proof of the Hodge theorem is related to the study of this operator. More precisely, $\alpha \in \Omega^k$ is *harmonic* if $\Delta\alpha = 0$. The Hodge decomposition for Riemannian manifolds states that within each de Rham cohomology class, there exists a unique harmonic representative.

In particular, this is a *canonical* representative for a cohomology class. Our goal is now to define harmonic metrics on flat vector bundles and on Higgs bundles, and this will be our tool to proving the non-abelian Hodge correspondence.

Recall the Riemann-Hilbert correspondence, which sends a representation $\rho : \pi_1(X) \to \mathbf{GL}(V)$ to the flat bundle $E = E_\rho$ given by:

$$E_\rho = \frac{\tilde{X} \times V}{\pi_1(X)}.$$

This has a natural flat connection, which we will denote by ∇. Under this correspondence, a subrepresentation corresponds to a subbundle which is preserved by the connection ∇.

Recall that a representation of a group G is *completely reducible* if it is a direct sum of irreducible representations. If G is finite and V is a finite dimensional vector space over $\mathbb{C}$, then Maschke's theorem implies that every representation is completely reducible. In our case, we would like to understand the completely reducible representations, as they will correspond to completely reducible flat bundles.

More concretely, we would like to find a sufficient condition for a flat bundle (E, ∇) to be completely reducible.

Choose a Hermitian metric h on E. Then given any subbundle $F \subseteq E$ which is ∇-invariant, we can then decompose:

$$\nabla_E = \begin{pmatrix} \nabla_F & \eta \\ 0 & \nabla_{F^\perp} \end{pmatrix},$$

where $\eta \in \Omega^1(X, \mathrm{Hom}(F^\perp, F))$ is the *second fundamental form*. Being reducible is equivalent to $\eta = 0$, and so we would like to look for a condition such that $\eta = 0$. Now decompose $\nabla = \mathrm{d}_A + \Psi$, where d_A is a unitary connection and Ψ is the Hermitian part of the connection. Locally, this is just writing an endomorphism as a sum of a skew-hermitian endomorphism and a hermitian endomorphism. More concretely:

$$\nabla_E = \begin{pmatrix} \mathrm{d}_F & \eta/2 \\ -\eta^*/2 & \mathrm{d}_{F^\perp} \end{pmatrix} + \begin{pmatrix} \Psi_F & \eta/2 \\ \eta^*/2 & \Psi_{F^\perp} \end{pmatrix}.$$

Consider the bundle $\mathrm{End}(E)$, and the section:

$$s = \begin{pmatrix} -I_F & 0 \\ 0 & I_{F^\perp} \end{pmatrix}.$$

Then:
$$\mathrm{d}_A s = \begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix},$$
and so:
$$\langle \Psi, \mathrm{d}_A s \rangle_{L^2} = \|\eta\|^2_{L^2}.$$

In particular, this means that a sufficient condition for $\eta = 0$ is that $\mathrm{d}_A^* \Psi = 0$.

Definition 3.7. A metric h on (E, ∇) is *harmonic* if $\mathrm{d}_A^* \Psi = 0$. We call (E, ∇, h) a *harmonic flat bundle.*

Remark 3.8. A priori, it may not be clear where the metric shows up in this definition. Note that both the decomposition $\nabla = \mathrm{d}_A + \Psi$ and the adjoint depends on the metric h. Moreover, the adjoint depends on a choice of metric on the base X.

Proposition 3.9. *A bundle (E, ∇, h) is a harmonic flat bundle if and only if:*
$$\begin{aligned} F(A) + \Psi \wedge \Psi &= 0, \\ \mathrm{d}_A \Psi &= 0, \\ \mathrm{d}_A^* \Psi &= 0. \end{aligned}$$

Theorem 3.10 ([**Don87**; **Cor88**])**.** *A flat bundle (E, ∇) admits a harmonic metric if and only if it is completely reducible, that is, its monodromy representation $\pi_1(X) \to \mathbf{GL}(n, \mathbb{C})$ is completely reducible. Moreover, the harmonic metric is unique up to scaling by positive constants.*

3.3.1 Harmonic functions between Riemannian manifolds

To be able to describe the proof of this theorem, we will need to discuss some theory of harmonic maps on manifolds.

Definition 3.11. Let (M_1, g_1) and (M_2, g_2) be oriented Riemannian manifolds. Let $f : M_1 \to M_2$ be a smooth map. The *energy* of f is:
$$E(f) = \int_{M_1} \|\mathrm{d}f\|^2 \mathrm{d}V_1$$
where $\mathrm{d}V_1$ is the Riemannian volume form with respect to g_1. We say that f is *harmonic* if f is a critical point of E.

Example 3.12. For example, take $M_1 = \mathbb{R}^n$ and $M_2 = \mathbb{R}$, with the usual Euclidean metrics. Then:
$$E(f) = \int_{\mathbb{R}^n} \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \right)^2 \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$
The associated Euler-Lagrange equation is:
$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i} \right) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} = \Delta f = 0,$$
and so this agrees with the usual definition of a harmonic function.

Let us now relate this definition to our definition of a harmonic metric. A Hermitian metric h on $\mathbb{C}^n$ is given by a positive definite hermitian (in the sense of its conjugate transpose being itself) matrix. We call the space of such metrics $\mathcal{H}$ and have a natural $G = \mathbf{GL}(n, \mathbb{C})$-action on $\mathcal{H}$ by:

$$g \cdot H = g^* H g.$$

This action is transitive, and the stabiliser of a point is isomorphic to $K = \mathbf{SU}(n)$. In particular, we have exhibited $\mathcal{H}$ as a symmetric space G/K.

Proposition 3.13. *The space of Hermitian metrics on $E = E_\rho$ can be identified with the space of $\pi_1(X)$-equivariant maps $u : \tilde{X} \to G/K$, where $\tilde{X}$ is the universal cover of X.*

The link between the two notions of harmonic is then given by the following lemma:

Lemma 3.14. *A map $u : \tilde{X} \to G/K$ is harmonic if and only if the associated metric h on (E, ∇) is harmonic. Moreover, changing the metric h by a gauge transformation is equivalent to changing u by a homotopy.*

To conclude this discussion, consider the following motivating theorem from the theory of harmonic maps between manifolds:

Theorem 3.15 ([**ES64**])**.** *Let M, N be compact Riemannian manifolds, where N has non-positive sectional curvature. Let $f : M \to N$ be a continuous map. Then f is homotopic to a smooth harmonic map $\tilde{f}$, which is the minimiser of the energy functional within the homotopy class.*

One proof of this theorem uses the direct method of the calculus of variations, and studies the lower-semicontinuity of the energy functional. This shows the existence of a weak solution, and then techniques in elliptic partial differential equations can be used to show that the weak solution is in fact smooth. See [**Jos17**, Chapter 9] for more details.

Note however that this theorem cannot be directly applied. The symmetric space G/K does have a natural non-positively curved metric. However, both $\tilde{X}$ (which is the hyperbolic plane) and G/K (which is diffeomorphic to $Lie(K)$) are non-compact.

3.3.2 Proof sketch of Theorem 3.10

As in the discussion following Theorem 2.8, the space $\mathcal{A}$ of connections on E is an affine space. We have a group action by $\mathcal{G} = \mathbf{SL}(E)$, and once we fix a metric h, we have a subgroup $\mathcal{K} = \mathbf{U}(E, h)$. Note that the groups here differ from the prior discussion, but not in an essential way. Our goal is to show that within a $\mathcal{K}$-orbit, there exists a unique harmonic representative. Here, we have taken the dual perspective of varying the connection, but the two perspectives are equivalent.

The original proof of Theorem 3.15 is via a non-linear heat flow. In [**Cor88**], Corlette adapts this proof to work in the context of flat bundles. The proof breaks down into the following steps:

1. Existence of a solution for small time t - this follows by an inverse function theorem argument.

2. Existence of a solution for all time t - this requires proving a priori estimates, showing that a solution on $[0, T)$ extends to a solution on $[0, T]$.

3. Convergence of the solution as $t \to \infty$ - the above argument also shows weak-boundedness of the solution, and so there is a weakly converging subsequence. By semicontinuity the limit is flat.

4. Showing that if the original bundle was stable, then the limiting connection must lie in the same gauge group orbit.

3.4 Harmonic Higgs bundles

We have an analogous story for harmonic Higgs bundles. To start, given a Higgs bundle (E, Φ), we would like to associate to it a flat connection ∇. To do this, fix a Hermitian metric h. Then we have the associated Chern connection ∇_A. Now consider:

$$\mathcal{A} = \Phi + \nabla_A + \Phi^*,$$

where we compute the adjoint Φ^* using the metric h. To explain this, we would like an equivalence between harmonic Higgs bundles and harmonic flat bundles. Given a harmonic flat bundle $(E, \nabla = \mathrm{d}_A + \Psi, h)$, we would like $\mathcal{A}$ to correspond to ∇. That is, $\Phi = \Psi^{1,0}$, and so $\Psi = \Phi + \Phi^*$.

Our goal is to find a Higgs bundle in the gauge orbit of (E, Φ), such that $\mathcal{A}$ defines a flat connection. Consider the $\mathbb{C}^*$-action given by:

$$\lambda \cdot (E, \Phi) = (E, \lambda\Phi).$$

Using this, we can consider the connections:

$$\mathcal{A}(\lambda) = \lambda\Phi + \nabla_A + \lambda^{-1}\Phi^*.$$

The curvature is then a Laurent polynomial in λ. More concretely, in a local trivialisation, we can write $\nabla_A = \mathrm{d} + A$, and we get that:

$$\begin{aligned} F(\mathcal{A}(\lambda)) &= \mathcal{A}(\lambda) \circ \mathcal{A}(\lambda), \\ &= \lambda^2 \Phi \wedge \Phi + \lambda^{-2}\Phi^* \wedge \Phi^*, \\ &+ \lambda(\bar{\partial} \circ \Phi + [A^{0,1}, \Phi]) + \lambda^{-1}(\partial \circ \Phi^* + [A^{1,0}, \Phi^*]), \\ &+ F(A) + [\Phi, \Phi^*]. \end{aligned}$$

The $\Phi \wedge \Phi$ term vanishes, as Φ is of type $(1, 0)$ and we are working over a curve. More generally, this is part of the definition of a Higgs bundle over higher-dimensional base spaces. Next, $\bar{\partial} \circ \Phi + [A^{0,1}, \Phi] = \bar{\partial}\Phi = 0$ as Φ is holomorphic. By taking hermitian conjugates, we get that the λ^{-2} and λ^{-1} coefficients vanishes as well. Thus, the only obstruction occurs in the λ^0-term. We have thus obtained the *Hitchin equation*:

$$F(A) + [\Phi, \Phi^*] = 0.$$

Definition 3.16. A *harmonic Higgs bundle* is a Higgs bundle (E, Φ), along with a Hermitian metric h, such that:

$$F(A) + [\Phi, \Phi^*] = 0.$$

Theorem 3.17 ([**Hit87**; **Sim88**])**.** *The Higgs bundle (E, Φ) is polystable if and only if within its complex gauge group orbit, there exists a unique (up to unitary gauge) harmonic representative.*

The idea of the proof is not too dissimilar from the proof of Theorem 3.10. The fact that existence of a harmonic representative implies polystability is the easier direction, and follows by a direct argument. For the converse, one defines a gradient flow, and studies the existence and convergence of the flow.

3.5 Non-abelian Hodge correspondence

With all of the set-up from above, we can now prove the non-abelian Hodge correspondence:

Proposition 3.18. *We have a one-to-one correspondence between harmonic flat bundles and harmonic Higgs bundles.*

Proof. Given a harmonic flat bundle $(E, \nabla = \mathrm{d}_A + \Psi, h)$, we can associate to it the Higgs bundle $(E, \nabla^{0,1} = \mathrm{d}_A^{0,1}, \Phi = \Psi^{0,1})$. In this case, the flatness condition:

$$F(A) + \Psi \wedge \Psi = 0$$

is equivalent to the Hitchin equation:

$$F(A) + [\Phi, \Phi^*] = 0.$$

Conversely, given a harmonic Higgs bundle $(E, \nabla^{0,1}, \Phi, h)$, we can associate to it the harmonic flat bundle $(E, \nabla = \Phi + \nabla_A + \Phi^*, h)$, where ∇_A is the Chern connection associated to $\nabla^{0,1}$ and h. It is easy to see that these two operations are inverse to each other. $\square$

Theorem 3.19 (non-abelian Hodge correspondence)**.** *The moduli space of polystable degree* 0 *Higgs bundles is in bijection with the character variety* $Rep^{cr}(\pi_1(X), \mathbf{GL}(n, \mathbb{C}))$ *of completely reducible representations.*

The proof of the theorem follows by combining the above proposition, Theorem 3.10 and Theorem 3.17.

3.6 Relation to Kobayashi-Hitchin correspondence

Let us now relate the non-abelian Hodge correspondence to the Kobayashi-Hitchin correspondence. If we set the Higgs field Φ to be zero, then the pair $(E, 0)$ is just a holomorphic vector bundle. It is easy to see that $(E, 0)$ is a stable Higgs bundle if and only if E is a stable vector bundle.

Setting $\Phi = 0$ also means that Hitchin's equation becomes $F(A) = 0$, which is the Hermite-Einstein equation for degree 0. Thus, we can think of the non-abelian Hodge correspondence as a generalisation of the Kobayashi-Hitchin correspondence.

Much like the Kobayashi-Hitchin correspondence can be considered as a higher-dimensional generalisation of the Narasimhan-Seshadri correspondence, it also makes sense to consider Higgs bundles and the non-abelian Hodge correspondence in higher dimensions. Throughout, fix a Kähler manifold (X, ω) of dimension n, and we will consider vector bundles on X.

The analogue of Theorem 3.10 in higher dimensions is:

Theorem 3.20. *A flat bundle E admits a harmonic metric if and only if it arises from a completely reducible representation.*

The analogue of Theorem 3.17, and a generalisation of Theorem 2.10 is:

Theorem 3.21. *A Higgs bundle (E, Φ) has a Hermite-Einstein metric if and only if it is polystable.*

Remark 3.22. One direction of this follows from the Kobayashi-Hitchin correspondence. For stability of Higgs bundles, we test against *fewer* subobjects compared to stability of holomorphic bundles. Thus, admitting a Hermite-Einstein metric implies polystability as a Higgs bundle follows from the Kobayashi-Hitchin correspondence. When $\Phi = 0$, this recovers the Kobayashi-Hitchin correspondence.

Proposition 3.23. *Let E be a holomorphic vector bundle on X, with a Hermite-Einstein metric h. The metric h is harmonic if and only if:*

$$ch_1(E) \cdot [\omega]^{n-1} = 0$$
$$ch_2(E) \cdot [\omega]^{n-2} = 0.$$

Remark 3.24. Note that the same ideas as before show that this is also equivalent to the associated Chern connection being flat. The first equation $ch_1(E) \cdot [\omega]^{n-1} = 0$ is the higher dimensional generalisation of $\deg(E) = 0$, as in this case, we need to choose a Kähler metric (or in the projective case, an ample line bundle). The second equation does not show up when X is a curve for degree reasons.

Using all of this, we can now state the general non-abelian Hodge correspondence:

Theorem 3.25. *Let (X, ω) be a compact Kähler manifold. Then there exists a bijection between:*

- *completely reducible representations of $\pi_1(X)$,*
- *polystable Higgs bundles E with:*

$$ch_1(E) \cdot [\omega]^{n-1} = 0$$
$$ch_2(E) \cdot [\omega]^{n-2} = 0.$$

3.7 Hitchin equation as a dimensional reduction of self-dual Yang-Mills

In the next section, we will see that the Higgs moduli spaces have a natural hyperkähler structure. To motivate this, we will now show that the Hitchin equation is a dimensional reduction of the self-dual (SD) Yang-Mills equations. Many examples of hyperkähler manifolds arise from (anti-)self-dual Yang-Mills, and so our study of harmonic Higgs bundles fits into this framework. By the non-abelian Hodge correspondence, this implies that we have a hyperkähler structure on the moduli space of stable Higgs bundles.

To be a bit more precise, the space of connections has a natural metric, given by the L^2-inner product. In many cases, this metric is in fact hyperkähler, and the moduli space which we are interested in can be obtained as a hyperkähler quotient. Note however that the spaces and groups involved here are infinite-dimensional, and so we cannot apply the theory of hyperkähler quotients directly.

Let (X, g) be a compact oriented Riemannian manifold, $E \to X$ a vector bundle with a fixed inner product. Then the *Yang-Mills functional* is:

$$A \mapsto \int_X \|F_A\|^2 \mathrm{d}V_g.$$

A connection is *Yang-Mills* if it is a critical point of this functional. The associated Euler-Lagrange equation is:

$$\nabla_A^* F_A = 0,$$

which can also be written as:

$$\nabla_A(*F_A) = 0.$$

Now suppose $\dim(X) = 4$. Then the Hodge star $\star : \Omega^2(X) \to \Omega^2(X)$ satsifies $\star^2 = 1$, and so we have a decomposition:

$$\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X)$$

of 2-forms into self-dual and anti-self-dual parts. Note that by the Bianchi identity, if $\star F_A = \pm F_A$, then $\mathrm{d}_A(\star F_A) = \pm \mathrm{d}_A F_A = 0$. Thus, we have obtained the *self-dual Yang-Mills equation*:

$$\star F_A = F_A.$$

Now consider the case of $X = \mathbb{R}^2 \times \mathbb{R}^2$, and E is a trivial bundle over X. We would like to consider connections invariant under translation of the second $\mathbb{R}^2$-factor. We can write any such connection as $A_0 \mathrm{d}x_0 + A_1 \mathrm{d}x_1 + \phi_0 \mathrm{d}y_0 + \phi_1 \mathrm{d}y_1$, where $A_0, A_1, \phi_0, \phi_1 : \mathbb{R}^2 \to \mathfrak{gl}(n, \mathbb{C})$ depends only on the (x_0, x_1)-coordinates. Let $z = x_0 + ix_1$, and define

$$\Phi = \frac{1}{2}(\phi_0 - i\phi_1)\mathrm{d}z.$$

In this case, the self-dual Yang-Mills equation becomes:

$$\begin{aligned} F_A + [\Phi, \Phi^*] &= 0, \\ \bar{\partial}_A \Phi &= 0, \end{aligned}$$

which are exactly the Hitchin equation and the definition of a Higgs bundle.

Note that in particular these equations are conformally invariant, and so makes sense on any Riemann surface, as we have seen above.

4 Properties of Higgs moduli spaces

4.1 Hyperkähler structure of Higgs moduli spaces

We wish to equip the Dolbeault moduli space $\mathcal{M}_{r,0}^{H,\mathrm{st}}(X)$ of rank r, degree 0, stable Higgs bundles with a hyperkähler structure. This structure encompasses the non-abelian Hodge correspondence in Theorem 3.5.

Recall the definition of a Kähler structure in Definition 1.8. Hyperkähler manifolds have three compatible Kähler structures:

Definition 4.1 (Hyperkähler manifolds)**.** Let M be a complex manifold with a Riemannian metric g, the tuple $(M, g, \omega_I, \omega_J, \omega_K)$ is a *hyperkähler manifold* if there are three Kähler structures:

- (M, g, ω_I, I),
- (M, g, ω_J, J),
- (M, g, ω_K, K),

such that their complex structures I, J, K fulfill the quaternionic relations:

$$I^2 = J^2 = K^2 = IJK = -\mathrm{id}_{TM}.$$

Note that the Kähler structures share the same metric g. For a treatment on hyperkähler manifolds and quotients, see for example [**HKLR87**; **Hit92**; **Joy07**; **Bes08**; **May19**].

4.1.1 Dolbeault structure

We first construct the Kähler structure on $(\mathcal{M}_{r,0}^{H,\mathrm{st}}(X), \omega_I)$. For this, we fix a degree zero smooth vector bundle $\mathbb{E}$ on X. A Higgs bundle (E, Φ) can be identified alternatively as the *Higgs pair* $(\nabla^{0,1}, \Phi)$ where $\nabla^{0,1}$ is a holomorphic structure $\nabla^{0,1}$, and $\Phi \in \Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$ is a Higgs field holomorphic with respect to $\nabla^{0,1}$. More precisely, we have $\nabla^{0,1}(\Phi) = 0 \in \Omega^{1,1}(X, \mathrm{End}(\mathbb{E}))$. The space of Higgs pairs $\mathcal{B}$ is a subspace of the affine space $\mathcal{A} \times \Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$:

$$\mathcal{B} = \{(\nabla^{0,1}, \Phi) | \nabla^{0,1}(\Phi) = 0\} \subseteq \mathcal{A} \times \Omega^{1,0}(X, \mathrm{End}(\mathbb{E})),$$

The real gauge group $\mathcal{G} = \mathrm{Aut}(\mathbb{E})$ acts on $\mathcal{A} \times \Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$ through conjugation, which restricts to $\mathcal{B}$.

We claim that there is a Kähler structure on $\mathcal{B}$, for which the action of $\mathcal{G}$ is Hamiltonian. For this, we fix a Hermitian metric h on $\mathbb{E}$, and obtain an isomorphism between holomorphic structures $\mathcal{A}$ and the space $\mathcal{A}(\mathbb{E}, h)$ of unitary connections preserving h, see our discussion in Section 3.4, and [**AB83**, page 570].

On $\Omega^1(X, \mathrm{End}(\mathbb{E}))$, we have an inner product given by:

$$(\alpha, \beta) = \int_X \alpha \wedge \star\beta, \text{ where } \star \text{ is the Hodge star,}$$

as seen in [**AB83**, page 547]. This is used to define an inner product g on $\mathcal{A}(\mathbb{E}, h)$, where for every tangent space $T_{d_A}\mathcal{A}(\mathbb{E}, h) \cong \Omega^{0,1}(X, \mathrm{Ad}(P))$ (P is the unitary bundle induced from $(\mathbb{E}, h)$), we have:

$$g_{d_A}(\alpha, \beta) = -\int_X \mathrm{tr}(\alpha \wedge \star\beta). \tag{4.1}$$

This inner product g can be used to induce a symplectic structure ω on $\mathcal{A}(\mathbb{E}, h)$, given by the symplectic form ω_{d_A} on $T_{d_A}\mathcal{A}(\mathbb{E}, h) \cong \Omega^{0,1}(X, \mathrm{Ad}(P))$:

$$\omega_{d_A}(\alpha, \beta) = -\int_X \mathrm{tr}(\alpha \wedge \beta), \text{ where } \alpha \wedge \beta \in \Omega^2(X, \mathrm{Ad}(P)),$$

as seen in [**McC18**, Proposition 4.4.1]. The definition of $\omega_{\nabla^{0,1}}$ does not depend on the fiber, so ω is a closed form. Through the isomorphism $\mathcal{A} \cong \mathcal{A}(\mathbb{E}, h)$, both the inner product and symplectic structure can be transferred to $\mathcal{A}$. These structures are compatible such that, along with the complex structure induced from $\mathcal{A}$ being a complex affine space, we have an (infinite-dimensional) Kähler structure on $\mathcal{A}$.

Similarly, on $\Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$, we have the symplectic structure given by the symplectic form ω_Φ on $T_\Phi\Omega^{1,0}(X, \mathrm{End}(\mathbb{E})) \cong \Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$:

$$\omega(\alpha, \beta) = -i\int_X \mathrm{tr}(\alpha \wedge \beta^*),$$

where β^* is the adjoint of β induced by the Hermitian metric h on $\mathbb{E}$ (see [**McC18**, 5.3.1]).

The symplectic structures we found give rise to a symplectic structure on $\mathcal{A} \times \Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$ that restricts to $\mathcal{B}$. In [**Tho23**, Theorem 5.1], we know that the $\mathcal{G}$-action of $\mathcal{A}$ induces a Hamiltonian reduction on $\mathcal{A}$. This extends to a Hamiltonian reduction on all of $\mathcal{A} \times \Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$, which restricts to $\mathcal{B}$.

The Hamiltonian reduction induces a symplectic structure on:

$$\mathcal{M}_{r,0}^{H,\mathrm{st}}(X) = \mathcal{B}/\mathcal{G}. \tag{4.2}$$

The space $\mathcal{B}$ is also a complex orbifold, as $\mathcal{B}$ projects to the second component $\Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$ which is a complex vector space, and $\mathcal{B}$ projects to the first component $\mathcal{A} \cong \mathcal{A}(\mathbb{E}, h)$, with an integrable complex structure through the Hodge star operator $\star$.

The symplectic structure and complex structure, with the Riemannian metric induced from (4.1), together are compatible such that they induce a Kähler structure on $\mathcal{M}_{r,0}^{H,\mathrm{st}}(X)$ called the *Dolbeault structure*.

To verify the equality in (4.2), where the left side is a Hamiltonian reduction, we need to look at the preimages of zero of the moment maps of the $\mathcal{G}$-action on $\mathcal{A}$ and $\Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$ respectively. As a reminder, we know that:

- The moment map of the $\mathcal{G}$-action on the space of unitary connections, compatible with h, maps a connection to its curvature: $\mu(\nabla) = F(\nabla) \in \mathrm{Lie}(\mathcal{G})^* \cong \Omega^2(X, \mathrm{End}(\mathbb{E}))$.
- The moment map of the $\mathcal{G}$-action on $\Omega^{1,0}(X, \mathrm{End}(\mathbb{E}))$ is $\mu(\Phi) = [\Phi, \Phi^*]$.
- The moment map of the $\mathcal{G}$-action on $\mathcal{B}$ (identifying $\mathcal{A}$ with unitary connections) is:

$$\mu((\nabla, \Phi)) = F(\nabla) + [\Phi, \Phi^*].$$

We saw the first moment map in the proof of Theorem 2.1, and the second one was discussed in Section 3.4. The final moment map combines the first two, which forms part of the *self-dual equations* from [**Hit87**], as seen in Section 3.4. The name was explained in Section 3.7, where we derived this from the self-dual Yang-Mills equations.

4.1.2 Betti structure

The *Betti structure* refers to the Kähler manifold $(\mathcal{M}_{r,0}^{H,\mathrm{st}}(X), \omega_J)$. Through the non-abelian Hodge correspondence in Chapter 3, we know that $\mathcal{M}_{r,0}^{H,\mathrm{st}}(X)$ is diffeomorphic to the character variety of irreducible representations:

$$\mathcal{M}_{r,0}^{H,\mathrm{st}}(X) \leftrightarrow \mathrm{Hom}^{\mathrm{irred}}(\pi_1(X), \mathbf{GL}(r, \mathbb{C}))/\mathbf{GL}(r, \mathbb{C}) \tag{4.3}$$

As a GIT-quotient, the character variety is an algebraic variety over $\mathbb{C}$, it's analytification has a Kähler structure that we lift to $\mathcal{M}_{r,0}^{H,\mathrm{st}}(X)$ through the non-abelian Hodge correspondence.

Remark 4.2. Note that I and J are *not* the same complex structure, the non-abelian Hodge correspondence will give us, at best, a diffeomorphism and not an isomorphism of complex analytic manifolds. As algebraic varieties, $\mathcal{M}_{r,0}^{H,\mathrm{st}}(X)$ is a non-affine quasi-projective variety, whereas $\mathrm{Hom}^{\mathrm{irred}}(\pi_1(X), \mathbf{GL}(r, \mathbb{C}))/\mathbf{GL}(r, \mathbb{C})$ is an affine variety.

The Riemann-Hilbert correspondence from Theorem 2.3 can be used to provide a complex analytic isomorphism from $\mathrm{Hom}^{\mathrm{irred}}(\pi_1(X), \mathbf{GL}(r, \mathbb{C}))/\mathbf{GL}(r, \mathbb{C})$ to $\mathcal{M}_{r,0}^{dR,\mathrm{st}}(X)$. Thus, the Betti structure can also be called the *de Rham structure*, as the Kähler structure on $\mathcal{M}_{r,0}^{dR,\mathrm{st}}(X)$ is also induced from flat-$\mathbf{GL}(r, \mathbb{C})$-connections on X.

Looking at Definition 4.1, we know $(\mathcal{M}_{r,0}^{H,\mathrm{st}}(X), \omega_I)$ and $(\mathcal{M}_{r,0}^{H,\mathrm{st}}(X), \omega_J)$. From this, the complex structure K can be determined due to the equality $IJK = -\mathrm{id}_{TM}$. Hence, we have equipped $\mathcal{M}_{r,0}^{H,\mathrm{st}}(X)$ with a hyperkähler structure.

4.2 The isosingularity theorem

From the non-abelian Hodge correspondence (Theorem 3.5), we have a bijection between polystable, rank r, degree 0, Higgs bundles, and the Betti moduli space of $\mathbf{GL}(r, \mathbb{C})$-representations. In this subsection, we will justify that this bijection is a diffeomorphism, using the *isosingularity theorem* from [**Sim94b**]:

Theorem 4.3 ([**Sim94b**, Theorem 10.6])**.** *For a $\mathbb{C}$-point $\alpha \in \mathcal{M}_{r,0}^{dR}(X)$ corresponding to a $\mathbb{C}$-point $\beta \in \mathcal{M}_{r,0}^{H}(X)$, there exists a canonical isomorphism between étale neighbourhoods of α and β.*

Let us unpack this, to make sense of étale neighbourhoods, we view the moduli spaces as schemes over $\mathbb{C}$. An étale neighbourhood (U, u) of α, as a $\mathbb{C}$-point, consists of an étale morphism of schemes $\varphi : U \to \mathcal{M}_{r,0}^{dR}(X)$ over $\mathbb{C}$, such that $\varphi(u) = \alpha$. Note that we cannot see U necessarily as a subscheme of $\mathcal{M}_{r,0}^{dR}(X)$.

Given étale neighbourhoods (U, u) of α and (V, v) of β, an isomorphism consists of an isomorphism of schemes $\Psi : U \to V$ such that $\Psi(u) = v$.

In particular, we are interested in *formal* étale neighbourhoods, i.e, for $\alpha \in \mathcal{M}_{r,0}^{dR}(X)$, and for the structure sheaf $\mathcal{O}^{dR}$ of $\mathcal{M}_{r,0}^{dR}(X)$, we complete the local ring $\mathcal{O}_\alpha^{dR}$ to obtain a formal étale neighbourhood (U, u) of α, where $U = \mathrm{Spf}(\widehat{\mathcal{O}_\alpha}) \to X$ is a (formally) étale map. The following lemma is an important tool that lets us derive complex analytic results from formal étale neighbourhoods:

Lemma 4.4. *Let Y be a complex algebraic variety with structure sheaf $\mathcal{O}_Y$. Let Y^{an} and $\mathcal{O}_Y^{an}$ be the analytification of Y and $\mathcal{O}_Y$ (as complex analytic manifolds). For a $\mathbb{C}$-point in Y, we have an isomorphism of local rings:*

$$\widehat{\mathcal{O}_y} \cong \widehat{\mathcal{O}_y^{an}}.$$

We can apply Theorem 4.3 as follows to the non-abelian Hodge correspondence:

Theorem 4.5. *There exists a diffeomorphism:*

$$\mathcal{M}_{r,0}^{H,pst}(X) \cong Hom(\pi_1(X), \mathbf{GL}(r, \mathbb{C}))/\mathbf{GL}(r, \mathbb{C})$$

between the moduli space of polystable degree 0 Higgs bundles and the $\mathbf{GL}(n, \mathbb{C})$*-character variety.*

For the proof, we will denote the structure sheaf of de Rham, Betti, and Dolbeault moduli spaces as $\mathcal{O}^{dR}$, $\mathcal{O}^{B}$, and $\mathcal{O}^{H}$, respectively.

Proof. In order to show that the bijection in Theorem 3.5 is a diffeomorphism, it is enough to show that analytic-locally at a point $\alpha \in \mathcal{M}_{r,0}^{H,\mathrm{pst}}(X)$, the bijection is smooth (as smoothness can be checked analytic-locally). It is thus enough to show that the correspondence is locally étale, i.e., the completions of local rings are isomorphic.

For a point $\alpha \in \mathcal{M}_{r,0}^{H,\mathrm{pst}}(X)$, we have the following isomorphisms of (completions of) local rings:

$$\widehat{\mathcal{O}_\alpha^B} \cong \widehat{\mathcal{O}_\alpha^{B,an}} \cong \widehat{\mathcal{O}_\alpha^{dR,an}} \cong \widehat{\mathcal{O}_\alpha^{dR}} \cong \widehat{\mathcal{O}_\alpha^{H}}.$$

The first and third isomorphisms come from Lemma 4.4, the second isomorphism comes from the Riemann-Hilbert correspondence in Theorem 2.3. The last isomorphism comes from the isosingularity theorem in Theorem 4.3. □

We now present a very rough proof sketch of the isosingularity theorem, following [**Sim94b**, Theorem 10.6]:

Proof sketch of Theorem 4.3.

- By the *Artin Approximation theorem*, it suffices to find the claimed isomorphism between formal étale neighbourhoods.
- For a fixed $x \in X$, we look at the representation spaces $\mathcal{R}_{r,0}^{dR}(X, x) \to \mathcal{M}_{r,0}^{dR}(X)$ and $\mathcal{R}_{r,0}^{DolB}(X, x) \to \mathcal{M}_{r,0}^{Dolb}(X)$ (see [**Sim94b**, pages 12, 17]).
- For a $\mathbb{C}$-point $\alpha \in \mathcal{M}_{r,0}^{dR}(X)$ and a $\mathbb{C}$-point $\beta \in \mathcal{M}_{r,0}^{H}(X)$, that lift to the same reductive representation $\rho : \pi_1(X, x) \to \mathbf{GL}(r, \mathbb{C})$ (through the last point), we have isomorphisms of coholomology rings:

$$H^{*,Dol}(X, \mathrm{Ad}(\rho)) \cong H^{*,dR}(X, \mathrm{Ad}(\rho)). \tag{4.4}$$

- Through this isomorphism, two cones controlling the deformation of α and β are isomorphic (see [**Sim94b**, Theorem 10.4]).
- The formal completions of the representation spaces are both isomorphic to the formal completion of this cone.

□

In [**Tir19**], Theorems 4.3 and 4.5 are used to show that the moduli space of Higgs bundles $\mathcal{M}_{r,0}^{H}(X)$ is represented by a complex algebraic variety that is a *symplectic singularity*, i.e., its smooth locus has a symplectic form, and any resolution of singularities is a *symplectic singularity*, i.e., the pullback of the symplectic form extends globally. Furthermore, $\mathcal{M}_{r,0}^{H}(X)$ admits projective symplectic resolutions exactly in the cases $g = 1$ or $(g, r) = (2, 2)$.

4.3 The twistor space and λ-connections

In this section, we will see an application of the diffeomorphism in Theorem 4.5 in terms of the hyperkähler manifold $\mathcal{M}_{r,0}^{H,\mathrm{st}}(X)$ we constructed in Section 4.1.

We can associate to a hyperkähler manifold $(M, \omega_I, \omega_J, \omega_K)$ its twistor space which combines all the three complex structures into one complex analytic manifold.

From the complex structures I, J, and K on M, we can parametrise a family of complex structures:

$$\kappa = \alpha I + \beta J + \gamma K, \quad \alpha^2 + \beta^2 + \gamma^2 = 1,$$

so we can interpret (α, β, γ) as a point in S^2 or $\mathbb{P}^1$.

Definition 4.6 (Twistor space)**.** For a hyperkähler manifold $(M, \omega_I, \omega_J, \omega_K)$, the *twistor space* $\mathrm{Tw}(M)$ is a complex analytic manifold with the underlying smooth manifold $M \times \mathbb{P}^1$. The complex structure on $\mathrm{Tw}(M)$ is such that the projection $p : \mathrm{Tw}(M) \to \mathbb{P}^1$ is holomorphic, and the fibre $p^{-1}(\alpha, \beta, \gamma)$, which is diffeomorphic to M, has the complex structure $\kappa = \alpha I + \beta J + \gamma K$.

Through the twistor space, the hyperkähler manifold $(\mathcal{M}_{r,0}^{H,\mathrm{st}}(X), \omega_I, \omega_J, \omega_K)$ unifies the Betti structure, de Rham, and Dolbeault structures. The twistor space allows us to *smoothly deform* between both structures through λ-connections, introduced by Deligne. The smooth deformation is possible due to Theorem 4.5, since the non-abelian Hodge correspondence is a diffeomorphism and not just a bijection.

Definition 4.7. For $\lambda \in \mathbb{C}$, a λ*-connection* ∇ on a complex vector bundle $\mathbb{E}$ on X is a map $\nabla : \Gamma(\mathbb{E}) \to \Gamma(T^*X \otimes \mathbb{E})$ that is $\mathbb{C}$-linear, and satisfies the Leibniz rule:

$$\nabla(fs) = f\nabla(s) + \lambda(df)s.$$

In letters from Deligne to Simpson, Deligne suggested the notion of *flat*-λ-connections to interpolate (in a diffeomorphic way) from flat connections to Higgs bundles. We fix $\mathbb{E} = X \times \mathbf{GL}(r, \mathbb{C})$ in the definition above. For $\lambda = 1$, we recover the usual notion of a flat connection. For $\lambda = 0$, we obtain a $C^\infty(X, \mathbb{C})$-linear operator, and so we recover a holomorphic structure from the unitary $(0,1)$-part (see Definition 1.3). From the $(1,0)$-part, we recover a Higgs field.

Viewing $\mathbb{P}^1$ as the Riemann sphere $\mathbb{C} \cup \{\infty\}$, the three structures I, J, K, correspond to points 0, 1, and ∞. In the affine patch $\mathbb{C} \subset \mathbb{C} \cup \{\infty\}$ around 0, each $\lambda \in \mathbb{C}$ induces a complex structure κ on $\mathcal{M}_{r,0}^{H,\mathrm{st}}(X)$ that corresponds to the *moduli space of flat-*λ*-connections* $\mathcal{M}_{r,0}^{\lambda}(X)$. Here, we have $\mathcal{M}_{r,0}^{1}(X) = \mathcal{M}_{r,0}^{dR}(X)$ and $\mathcal{M}_{r,0}^{0}(X) = \mathcal{M}_{r,0}^{H}(X)$ (with the Dolbeault structure) (see [**Sim96**, Proposition 4.1]).

Definition 4.8 (Hodge moduli space)**.** The *Hodge moduli space* $\mathcal{M}_{r,0}^{\mathrm{Hod}}(X)$ is the moduli space of flat-λ-connections on X, for varying $\lambda \in \mathbb{C}$.

The complex analytic manifold $\mathcal{M}_{r,0}^{\mathrm{Hod}}(X)$ comes equipped with a map $\mathcal{M}_{r,0}^{\mathrm{Hod}}(X) \to \mathbb{C}$, mapping a λ-connection to the value $\lambda \in \mathbb{C}$. $\mathcal{M}_{r,0}^{\mathrm{Hod}}(X) \to \mathbb{C}$ is a smooth fibre bundle that has moduli spaces of λ-connections as fibres, particularly the fibres $\mathcal{M}_{r,0}^{dR}(X)$ and $\mathcal{M}_{r,0}^{H}(X)$.

We now explain that we can smoothly glue two copies of the de Rham moduli space to obtain the Hodge moduli space $\mathcal{M}_{r,0}^{\mathrm{Hod}}(X)$. By flipping the orientation of X, we obtain a Riemann surface $\overline{X}$ with a conjugate complex structure to X. We claim that we can smoothly glue $\mathcal{M}_{r,0}^{dR}(X)$ and $\mathcal{M}_{r,0}^{dR}(\overline{X})$ together to form a bundle $\mathcal{M}_{r,0}^{DH}(X)$ over $\mathbb{P}^1$, called the *Deligne-Hitchin twistor space.*

We use the Riemann sphere perspective $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$ to glue $\mathcal{M}_{r,0}^{dR}(X)$ and $\mathcal{M}_{r,0}^{dR}(\overline{X})$. On the affine patch $\mathbb{C} \subset \mathbb{C} \cup \{\infty\}$ around 0, $\mathcal{M}_{r,0}^{DH}(X)$ restricted to $\mathbb{C}$ is defined as $\mathcal{M}_{r,0}^{dR}(X) \to \mathbb{C}$. On the affine patch $\mathbb{C}^* \cup \{\infty\}$ around ∞, $\mathcal{M}_{r,0}^{DH}(X)$ restricted to $\mathbb{C}^* \cup \{\infty\}$ is defined as $\mathcal{M}_{r,0}^{dR}(\overline{X}) \to \mathbb{C}^* \cup \{\infty\}$, mapping a λ-connection on $\overline{X}$ to λ^{-1} (using the convention $1/0 = \infty$). In the intersection of the two charts, we glue by identifying a λ-connection of $\mathcal{M}_{r,0}^{DH}(X)$ with a λ^{-1}-connection of $\mathcal{M}_{r,0}^{DH}(\overline{X})$. This gluing is complex analytic, but not algebraic, as it uses the Riemann-Hilbert correspondence. See [**Sim96**, Proposition 4.1, Remark] for more.

This leads to the following isomorphism:

Theorem 4.9 ([**Sim96**]). *$\mathcal{M}_{r,0}^{DH}(X)$ is isomorphic to $\mathcal{M}_{r,0}^{Hod}(X)$, as complex analytic bundles over $\mathbb{P}^1$.*

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